

**A COMMENTARY ON TEICHMÜLLER'S PAPER  
VERSCHIEBUNGSSATZ DER QUASIKONFORMEN ABBILDUNG  
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**ABSTRACT.** This is a commentary on Teichmüller's paper *Ein Verschiebungssatz der quasikonformen Abbildung* (A displacement theorem of quasiconformal mapping), published in 1944. We explain in detail how Teichmüller solves the problem of finding the quasiconformal mapping from the unit disc to itself, sending 0 to a strictly negative point on the real line, holding the boundary of the disc pointwise fixed and with the smallest quasiconformal dilatation. We mention also some consequences of this extremal problem and we ask a question.

## 1. INTRODUCTION

This is a commentary on Teichmüller's paper *Ein Verschiebungssatz der quasikonformen Abbildung*, published in 1944<sup>1</sup> (see [30]). We refer to the English translation which appears in this volume. The present paper is part of a series of commentaries written by various authors on papers of Teichmüller. These papers contain some ideas which are still unknown to Teichmüller theorists, see for example [2], [6] and [1]. The paper [30] is one of the last that Teichmüller wrote, and especially the last one about quasiconformal maps. In this paper, he solved the following geometric problem:

**Problem 1.1.** Find and describe the quasiconformal map from the unit disc to itself such that

- its restriction to the unit circle is the identity map,
- the image of 0 is  $-x$ , where  $0 < x < 1$ ,
- its quasiconformal dilatation is as small as possible.

As Teichmüller wrote at the beginning of his paper, this extremal problem is rather different from those studied in [28]. It is due to the fact that mappings fix all boundary points and not only a finite number of such points. The paper [28] is at the foundation of the theory that we call now the *classical Teichmüller theory*.

In order to solve Problem 1.1, Teichmüller used an idea already contained in §23 and §24 of [28].<sup>2</sup> Indeed, he obtained an equivalent problem (see Problem 4.1 in Subsection 4.1) by taking ramified coverings, which turns out to be simpler. Let us say a few words about that. First, he constructed, using explicit conformal maps, two 2-sheeted branched coverings of the unit disc, the first one branched at 0 and

<sup>1</sup>Note that this paper appeared after his death which occurred in 1943 on the Eastern Front.

<sup>2</sup>Even if the idea was already used, the result was not known from specialists. We refer to [16] and especially to what Grötzsch told to Kühnau about this paper: "Ja...ah, das habe ich nicht gehabt [Okay, this I did not have]."

the other at  $-x$ . These two covering spaces can be conformally represented by ellipses with data depending on  $x$ . This construction shows that the main problem is equivalent to a problem of minimization of the quasiconformal dilatation for mappings between two ellipses with a particular condition on the boundary. After that, he showed, using the Cauchy-Schwarz inequality (as in the solution to the Grötzsch problem), that the extremal map between the ellipses is given by an affine transformation. Finally, he gave for the quasiconformal dilatation of the extremal map a lower bound depending on  $x$  and an asymptotic behaviour when  $x$  approaches 0.

In [30], Teichmüller did not give definitions; all the definitions he used are in [28]. This is why in these notes we will change the organization of the text in comparison with [30]; but we will keep all the ideas from Teichmüller.

After recalling some notation and definitions, especially about quasiconformal mappings, we will explain the Grötzsch problem and we will recall the notion of Grötzsch module. We will then give details on the proof of Teichmüller. We will conclude by some applications of this result.

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## 2. PRELIMINARIES

All along this paper, we shall be interested in planar quasiconformal mappings. Unless otherwise noted, all domains that we consider are connected subsets of the Riemann sphere  $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ . There are several books which deal with quasiconformal mappings, see e.g. [5], [17] or [8].

We give below two equivalent definitions of quasiconformal maps. Both of them are interesting and they introduce notions (module and quasiconformal dilatation) that Teichmüller used to solve Problem 1.1.

A *quadrilateral*  $\mathcal{Q}$  is a Jordan domain (i.e. a simply connected domain whose boundary is a Jordan curve) with four distinct boundary points. Sometimes, we will denote by  $\mathcal{Q}(a, b, c, d)$  such a quadrilateral, where  $a, b, c$  and  $d$  are boundary points and we shall usually assume that these four points appear on the boundary in that order. By applying successively the Riemann Mapping Theorem, the Carathéodory Theorem<sup>3</sup> and a suitable Schwarz-Christoffel mapping, we know that  $\mathcal{Q}$  is conformally equivalent (i.e. there exists a holomorphic bijection) to a rectangle

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<sup>3</sup>The theorem referred to is known as the *boundary correspondance theorem*.

$\mathcal{R}$  of vertical side length 1 and horizontal length side  $m$ , for some uniquely defined  $m > 0$ .<sup>4</sup> We call the *module* of  $\mathcal{Q}$ , denoted by  $\text{mod}(\mathcal{Q})$ , the number  $m$ .

A *doubly-connected domain*  $\mathcal{C}$  is a connected domain whose boundary is the union of two disjoint Jordan curves. As for the quadrilateral, we can associate a module to a doubly-connected domain. We know that such a domain is conformally equivalent to an annulus whose inner radius is 1 and outer radius is  $R$ , for some  $R > 1$ . We call the *module* of  $\mathcal{C}$ , denoted by  $\text{mod}(\mathcal{C})$ , the number  $\frac{1}{2\pi} \log(R)$ .

**Remark 2.1.** Another way to introduce the module is to define it as the inverse of *extremal length* of a particular family of curves. This relation enables us to extend the notion of quasiconformal mapping in higher dimensions.

**Definition 2.2** (Geometric definition). Let  $\Omega$  be an open set in  $\mathbb{C}$ . Let  $f : \Omega \rightarrow f(\Omega)$  be an orientation-preserving homeomorphism. We say that  $f$  is *quasiconformal* if there exists  $K \geq 1$  such that for any quadrilateral  $\mathcal{Q} \subset \Omega$ ,

$$\text{mod}(f(\mathcal{Q})) \leq K \cdot \text{mod}(\mathcal{Q}).$$

In this case, we set  $K_f := \sup_{\mathcal{Q}} \frac{\text{mod}(f(\mathcal{Q}))}{\text{mod}(\mathcal{Q})}$  and we call it the *quasiconformal dilatation*<sup>5</sup> of  $f$ . Moreover, we say that  $f$  is  $K_f$ -quasiconformal.

To simplify notation, we write *q.c.* instead of quasiconformal.

With this definition, it is easy to see that for  $f_1$  and  $f_2$  respectively  $K_1$ -q.c. and  $K_2$ -q.c. on suitable domains,  $f_1 \circ f_2$  is  $K_1 K_2$ -q.c.

We can show that  $f$  is conformal if and only if  $K = 1$ . Thus, if  $g$  and  $h$  are conformal, then  $g \circ f \circ h$  has the same q.c. dilatation as  $f$ .

Before giving an equivalent definition of q.c. mappings, we recall that a map  $f$  is *absolutely continuous on lines* (ACL) in a domain  $\Omega$  if for every rectangle  $R := \{x + iy \mid a < x < b, c < y < d\}$  in  $\Omega$ ,  $f$  is absolutely continuous as a function of  $x$  (resp.  $y$ ) on almost all segments  $I_y := \{x + iy \mid a < x < b\}$  (resp.  $I_x := \{x + iy \mid c < y < d\}$ ). We can show that such a function  $f$  is differentiable almost everywhere (a.e.) in  $\Omega$ .

The second equivalent definition of quasiconformality is the following.

**Definition 2.3** (Analytic definition). Let  $\Omega$  be an open set of  $\mathbb{C}$ . Let  $f : \Omega \rightarrow f(\Omega)$  be a homeomorphism. We say that  $f$  is  $K$ -quasiconformal if

- (1)  $f$  is ACL on  $\Omega$ ,
- (2)  $|\partial_{\bar{z}} f| \leq k \cdot |\partial_z f|$  (a.e), where  $k = \frac{K-1}{K+1}$ .

<sup>4</sup>To be more precise, the interior of  $\mathcal{Q}$  is sent (conformally) onto the upper half-plane  $\mathbb{H}$  and this map can be extended to a homeomorphism from  $\mathcal{Q}$  to  $\mathbb{H} \cup \mathbb{R} \cup \{\infty\}$ . Moreover, the four distinguished points of  $\mathcal{Q}$  are sent respectively to 0, 1,  $\lambda$  and  $\infty$  for some  $\lambda > 1$ . Finally,  $z \mapsto c \int_z \frac{\zeta}{\sqrt{\zeta(\zeta-1)(\zeta-\lambda)}}$  (for a suitable  $c$ ) maps the upper half-plane onto the rectangle  $\mathcal{R}$  of vertical length side 1 and horizontal length side  $m$ .

<sup>5</sup>This is not the term that Teichmüller used in his papers (for example [28], [30] and [29]). He used the term “dilatation quotient.” In the current literature, we can also find the terms “maximal dilatation,” “dilatation,” “distorsion” or “quasiconformal norm.”

We recall that

$$\begin{cases} \partial_z f &= \frac{1}{2} (\partial_x f - i \partial_y f), \\ \partial_{\bar{z}} f &= \frac{1}{2} (\partial_x f + i \partial_y f), \\ \text{Jac}(f) &= |\partial_z f|^2 - |\partial_{\bar{z}} f|^2. \end{cases}$$

For a q.c. mapping  $f$  (in the sense of Definition 2.3), we can show that

$$K_f = \text{ess. sup}_{z \in \Omega} \frac{|\partial_z f(z)| + |\partial_{\bar{z}} f(z)|}{|\partial_z f(z)| - |\partial_{\bar{z}} f(z)|}.$$

According to the introduction of [17], Definition 2.3 was introduced by Morrey in [21]. Moreover, Definition 2.2 is due to Ahlfors (see [4]). Works by Bers, Mori and Yûjôbô<sup>6</sup> show that Definition 2.2 and Definition 2.3 are equivalent. A proof can be found in [5].

Grötzsch introduced q.c. mappings in [10] under the name “nichtkonforme Abbildungen.”<sup>7</sup> His definition is similar to Definition 2.3 but with a stronger hypothesis: he assumes the maps differentiable. Teichmüller was the first to use q.c. mappings in a substantial manner. He developed this theory to study the *Teichmüller space*<sup>8</sup> and the so-called *Teichmüller metric*. Note that he used in [28], and [30] the Grötzsch definition for q.c. mappings in a slightly different form. He considered mappings which are differentiable “up to finitely many closed analytical curve segments.” Since nothing changes in Teichmüller’s proof (Proposition 4.2 below) we shall use Definition 2.3 for q.c. mappings. In particular, using the point of view of Definition 2.3, we can show that for any q.c. mapping  $f$ ,  $\partial_z f \neq 0$  (a.e.).

### 3. GRÖTZSCH’S PROBLEM AND GRÖTZSCH’S DOMAIN

**3.1. Grötzsch’s problem.** To conclude the solution of Problem 1.1, Teichmüller used the same idea as Grötzsch used in [13] to solve what we call now the Grötzsch problem. In fact, Teichmüller used a kind of generalization of Grötzsch’s problem in his development of classical Teichmüller theory. So, let us present this problem.

Let  $\mathcal{R}_1$  and  $\mathcal{R}_2$  be two quadrilaterals. As already mentioned at the beginning of Section 2, we can suppose that  $\mathcal{R}_1$  and  $\mathcal{R}_2$  are rectangles whose modules are respectively  $a_1$  and  $a_2$  (see Figure 1).

The *Grötzsch problem* is the following:

*Find and describe the q.c. mapping from  $\mathcal{R}_1$  to  $\mathcal{R}_2$  which preserves sides and with the smallest q.c. dilatation.*

First, we have to find a q.c. mapping from  $\mathcal{R}_1$  to  $\mathcal{R}_2$ . The simplest map is the following

$$z \mapsto \frac{1}{2} \left( 1 + \frac{a_2}{a_1} \right) \cdot z + \frac{1}{2} \left( \frac{a_2}{a_1} - 1 \right) \cdot \bar{z},$$

<sup>6</sup>This list is not exhaustive.

<sup>7</sup>The name *quasiconformal* is now credited to Ahlfors, but this is debatable (see the commentary related to [3] p. 213).

<sup>8</sup>Teichmüller called it “the space of topologically determined principal regions.”

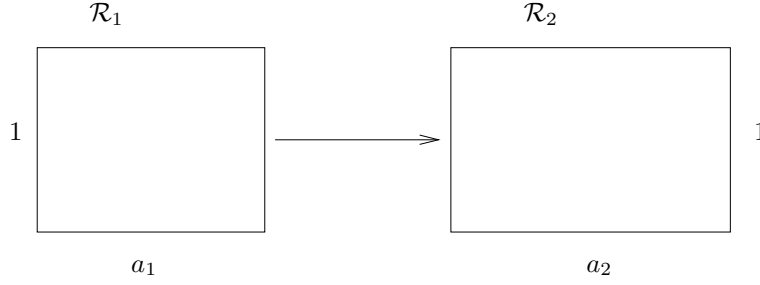


FIGURE 1.

and its q.c. dilatation is equal to  $\max\left(\frac{a_1}{a_2}, \frac{a_2}{a_1}\right)$ . By the Cauchy-Schwarz inequality, we check that if  $f$  is a q.c. mapping between these two rectangles, then  $K_f \geq \max\left(\frac{a_1}{a_2}, \frac{a_2}{a_1}\right)$ , with equality if and only if  $f$  is the affine map above. This solves this problem.

Teichmüller used the same principle. He found a q.c. mapping which can be a candidate and he showed in the same manner that this map is the one with the smallest q.c. dilatation.

**3.2. Grötzsch's domain and its associated module.** A Grötzsch domain is an extremal domain for the following problem. Let  $R > 1$  be a real number and let  $\Omega \subset \mathbb{C}$  be a doubly-connected domain separating the unit circle  $\partial\mathbb{D}$  from  $\{R, \infty\}$ . Such a domain has a module and we want to know whether there exists a domain whose module is maximal. The answer is yes, and we can describe it.

This domain, now called a *Grötzsch domain*, is  $\mathbb{C} \setminus (\overline{\mathbb{D}} \cup [R, \infty))$ . We denote its module by  $\frac{1}{2\pi} \log(\Phi(R))$ . Before Teichmüller, some facts about the map  $\Phi$  were already known to Grötzsch (see [11, 12]), like the fact that  $\Phi : ]1, +\infty[ \rightarrow ]1, +\infty[$  is an increasing continuous function such that

$$(3.1) \quad \forall R > 1; \quad R < \Phi(R) < 4R,$$

and

$$(3.2) \quad \lim_{R \rightarrow \infty} (\Phi(R) - 4R) = 0.$$

According to [30], Teichmüller “proved in a *purely geometric way*” properties (3.1) and (3.2). See [27] for these proofs.

We give below a functional relation with a sketch of proof. Let  $\alpha$  be a positive number strictly less than 1. The doubly-connected domain  $\mathbb{D} \setminus ([-\alpha, \alpha])$  has a module which is (with our notation)

$$\frac{1}{2\pi} \log \left( \Phi \left( \frac{1}{2} \left( \alpha + \frac{1}{\alpha} \right) \right) \right).$$

Indeed, the image of  $\mathbb{D} \setminus ([-\alpha, \alpha])$  by the biholomorphism of  $\mathbb{D}$  sending  $-\alpha$  to 0 and  $\alpha$  to  $\frac{2\alpha}{1+\alpha^2}$  is  $\mathbb{D} \setminus \left[0, \frac{2\alpha}{1+\alpha^2}\right]$ . By applying  $z \mapsto 1/z$ , we see that its module is exactly what we wrote. Moreover, the Grötzsch domain associated with  $1/\alpha^2$  is

equivalent to

$$\mathbb{C} \setminus \left( [-1, 1] \cup \left[ \frac{1}{2} \left( \alpha^2 + \frac{1}{\alpha^2} \right), \infty \right) \right).$$

To see this, we use the map  $z \mapsto \frac{1}{2} \left( z + \frac{1}{z} \right)$ . Note that this map will be important in the solution of our problem and also that it is the bridge between the Grötzsch domain and what is now called the *Teichmüller domain* (see Chapter 3 of [5]). Now by  $z \mapsto \frac{\alpha}{1 + \alpha^2} (z + 1)$ , we reach

$$\mathbb{C} \setminus \left( \left[ 0, \left( \frac{1}{2} \left( \alpha + \frac{1}{\alpha} \right) \right)^{-1} \right] \cup \left[ \frac{1}{2} \left( \alpha + \frac{1}{\alpha} \right), \infty \right) \right).$$

This domain has, by reflection with respect to the unit circle, a module equal to

$$\frac{1}{\pi} \log \left( \Phi \left( \frac{1}{2} \left( \alpha + \frac{1}{\alpha} \right) \right) \right).$$

Thus, we obtain the following relation

$$(3.3) \quad \Phi \left( \frac{1}{2} \left( \alpha + \frac{1}{\alpha} \right) \right) = \sqrt{\Phi \left( \frac{1}{\alpha^2} \right)}.$$

The expressions “Grötzsch’s domain” and “Teichmüller’s domain” are used by Ahlfors in [5] and also by Lehto and Virtanen in [17] whose German version was published in 1965. The author of this report does not know who was the first person to use this terminology.

#### 4. THE SOLUTION OF TEICHMÜLLER’S PROBLEM

**4.1. A simple case.** For two strictly positive real numbers  $\alpha$  and  $\beta$ , we denote by  $\mathcal{E}(\alpha, \beta)$  the ellipse whose centre is the origin and the major (resp. minor) axis is equal to  $\alpha$  (resp.  $\beta$ ). We want to solve the following extremal problem:

**Problem 4.1.** Is there a q.c. mapping from  $\mathcal{E}(\alpha, \beta)$  to  $\mathcal{E}(\beta, \alpha)$  with the smallest q.c. dilatation and whose restriction to the boundary coincides with the restriction of  $h_0 : z \mapsto \frac{1}{2} \left( \frac{\alpha}{\beta} + \frac{\beta}{\alpha} \right) \cdot z + \frac{1}{2} \left( \frac{\alpha}{\beta} - \frac{\beta}{\alpha} \right) \cdot \bar{z}$ ?

It is easy to show that  $h_0$  sends  $\mathcal{E}(\alpha, \beta)$  to  $\mathcal{E}(\beta, \alpha)$  with the good behaviour at the boundary. Moreover, its q.c. dilatation is equal to  $\max \left( \frac{\alpha^2}{\beta^2}, \frac{\beta^2}{\alpha^2} \right)$ . We will show that this is the smallest q.c. dilatation with the given conditions.

**Proposition 4.2.** *Let  $f : \mathcal{E}(\alpha, \beta) \mapsto \mathcal{E}(\beta, \alpha)$  be a q.c. mapping such that  $f|_{\partial \mathcal{E}(\beta, \alpha)}$  coincides with the restriction of  $h_0$ . Then,*

$$K_f \geq \max \left( \frac{\alpha^2}{\beta^2}, \frac{\beta^2}{\alpha^2} \right).$$

*Proof.* As in the solution of the Grötzsch problem, we can suppose that  $f$  is continuously differentiable in both directions. Let  $y \in ]-\beta, \beta[$ . We denote by  $l(y)$  the Euclidean length of the segment  $\text{Im}(z) = y$  in  $\mathcal{E}(\alpha, \beta)$ . We parametrize this

segment by  $\gamma_y : t \in \left[-\frac{l(y)}{2}, \frac{l(y)}{2}\right]$ . Due to the hypothesis on  $f|_{\partial\mathcal{E}(\beta, \alpha)}$ , the length of  $f \circ \gamma_y$  is bigger than  $\frac{\alpha}{\beta} \cdot l(y)$ . We have the following inequality

$$(4.4) \quad \frac{\alpha}{\beta} \cdot l(y) \leq \int_{-\frac{l(y)}{2}}^{\frac{l(y)}{2}} |(f \circ \gamma_y)'(t)| dt.$$

But

$$(f \circ \gamma_y)'(t) = \partial_z f(\gamma_y(t)) \cdot \gamma_y'(t) + \partial_{\bar{z}} f(\gamma_y(t)) \cdot \overline{\gamma_y'(t)},$$

so

$$(4.5) \quad \begin{aligned} |(f \circ \gamma_y)'(t)| &\leq (|\partial_z f(\gamma_y(t))| + |\partial_{\bar{z}} f(\gamma_y(t))|) \cdot |\gamma_y'(t)| \\ &= \left( \frac{|\partial_z f(\gamma_y(t))| + |\partial_{\bar{z}} f(\gamma_y(t))|}{|\partial_z f(\gamma_y(t))| - |\partial_{\bar{z}} f(\gamma_y(t))|} \cdot (|\partial_z f(\gamma_y(t))|^2 - |\partial_{\bar{z}} f(\gamma_y(t))|^2) \right)^{\frac{1}{2}} \\ &\leq (K_f \cdot \text{Jac}(f)(\gamma_y(t)))^{\frac{1}{2}}. \end{aligned}$$

By applying the Cauchy-Schwarz inequality in (4.4) and using (4.5), we obtain

$$\frac{\alpha^2}{\beta^2} \cdot l(y) \leq K_f \int_{-\frac{l(y)}{2}}^{\frac{l(y)}{2}} \text{Jac}(f)(\gamma_y(t)) dt.$$

Integration with respect to  $y$  gives us

$$\frac{\alpha^2}{\beta^2} \leq K_f.$$

If we replace the horizontal segment by the vertical segment in the ellipse, by the same method we obtain

$$\frac{\beta^2}{\alpha^2} \leq K_f,$$

and so, the proof is complete.  $\square$

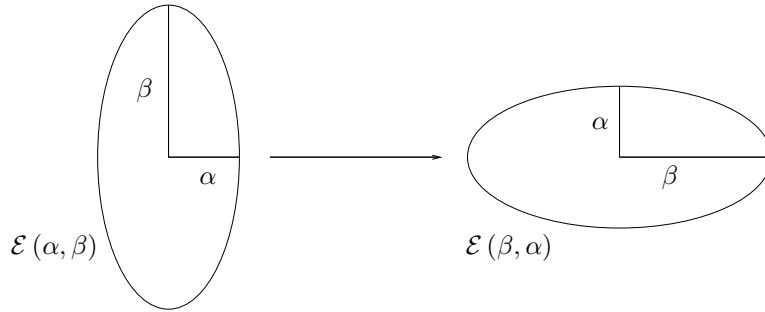


FIGURE 2. Like in the case of rectangles, we are looking for the q.c. mapping with the smallest q.c. dilatation and a good behaviour at the boundary.

**4.2. Solution.** We will explain how Teichmüller showed that Problem 1.1 is equivalent to Problem 4.1 for some  $(\alpha, \beta)$  that we shall specify.

We start with  $\overline{\mathbb{D}} \setminus [-x, 0]$ . Its preimage by the covering map,  $z \mapsto z^2$  is  $\overline{\mathbb{D}} \setminus i[-\sqrt{x}, \sqrt{x}]$ . The interior of the latter domain can be conformally sent by  $\varphi$  onto an annulus of inner radius 1 and outer radius  $R$ . To see this, we have to map (conformally) the first quadrant of the unit disc onto the first quadrant of this annulus for some  $R$ . Such a map exists by the same arguments given in Section 2. By successive reflections with respect to the horizontal and the vertical axes, we can define  $\varphi$ . Note that according to (3.3),

$$(4.6) \quad R = \sqrt{\Phi\left(\frac{1}{x}\right)}.$$

We must say that Teichmüller used a nicer method to obtain (4.6). Indeed, this relation is given by the following commutative diagram

$$\begin{array}{ccc} \mathbb{D} \setminus i[-\sqrt{x}, \sqrt{x}] & \xrightarrow{\varphi} & \mathcal{C}(1, R) \\ \downarrow z \mapsto z^2 & & \downarrow \varphi \\ \mathbb{D} \setminus [-x, x] & \xrightarrow{\quad} & \mathcal{C}\left(1, \Phi\left(\frac{1}{x}\right)\right) \end{array}$$

where  $\mathcal{C}(1, R)$  (resp.  $\mathcal{C}\left(1, \Phi\left(\frac{1}{x}\right)\right)$ ) denotes the annulus whose inner radius is 1 and outer radius is  $R$  (resp.  $\Phi\left(\frac{1}{x}\right)$ ).

Finally,  $f_1 : z \mapsto z - \frac{1}{z}$  and  $f_2 : z \mapsto z + \frac{1}{z}$  map the annulus  $\mathcal{C}(1, R)$  onto  $\mathcal{E}\left(R - \frac{1}{R}, R + \frac{1}{R}\right) \setminus i[-2, 2]$  and  $\mathcal{E}\left(R + \frac{1}{R}, R - \frac{1}{R}\right) \setminus [-2, 2]$  respectively. To simplify notation, we set  $\mathcal{E}_1 := \mathcal{E}\left(R - \frac{1}{R}, R + \frac{1}{R}\right)$  and  $\mathcal{E}_2 := \mathcal{E}\left(R + \frac{1}{R}, R - \frac{1}{R}\right)$ .

Thus, we have two new maps,  $p_1 := (\varphi^{-1} \circ f_1^{-1})^2$  and  $p_2 := (\varphi^{-1} \circ f_2^{-1})^2$ . The mapping  $p_1$  (resp.  $p_2$ ) can be extended to a map from  $\mathcal{E}_1$  (resp.  $\mathcal{E}_2$ ) to  $\mathbb{D}$  such that 0 is sent to 0 (resp.  $-x$ ). We denote the associated maps again by  $p_1$  and  $p_2$ . For more details, see Figure 3. Note that in Teichmüller's paper [30], there is an equivalent figure.

Now, we remark that  $p_1 : \mathcal{E}_1 \rightarrow \mathbb{D}$  (resp.  $p_2 : \mathcal{E}_2 \rightarrow \mathbb{D}$ ) is a two-sheeted ramified covering, where the branch point is 0 (resp.  $-x$ ).

We now have all the elements to solve our problem. We start by recalling the problem. Let  $f$  be a q.c. mapping from  $\mathbb{D}$  to  $\mathbb{D}$  such that  $f(0) = -x$  and  $f|_{\partial\mathbb{D}} = \text{id}_{\partial\mathbb{D}}$ . Since  $f$  maps the branch point 0 to the branch point  $-x$ , we can lift it. We denote this lift by  $\tilde{f}$  (see Figure 4). It is easy to check that  $\tilde{f} : \mathcal{E}_1 \rightarrow \mathcal{E}_2$  is a q.c. mapping, with the same q.c. dilatation as  $f$ . Furthermore,  $\tilde{f}(0) = 0$  and its restriction to



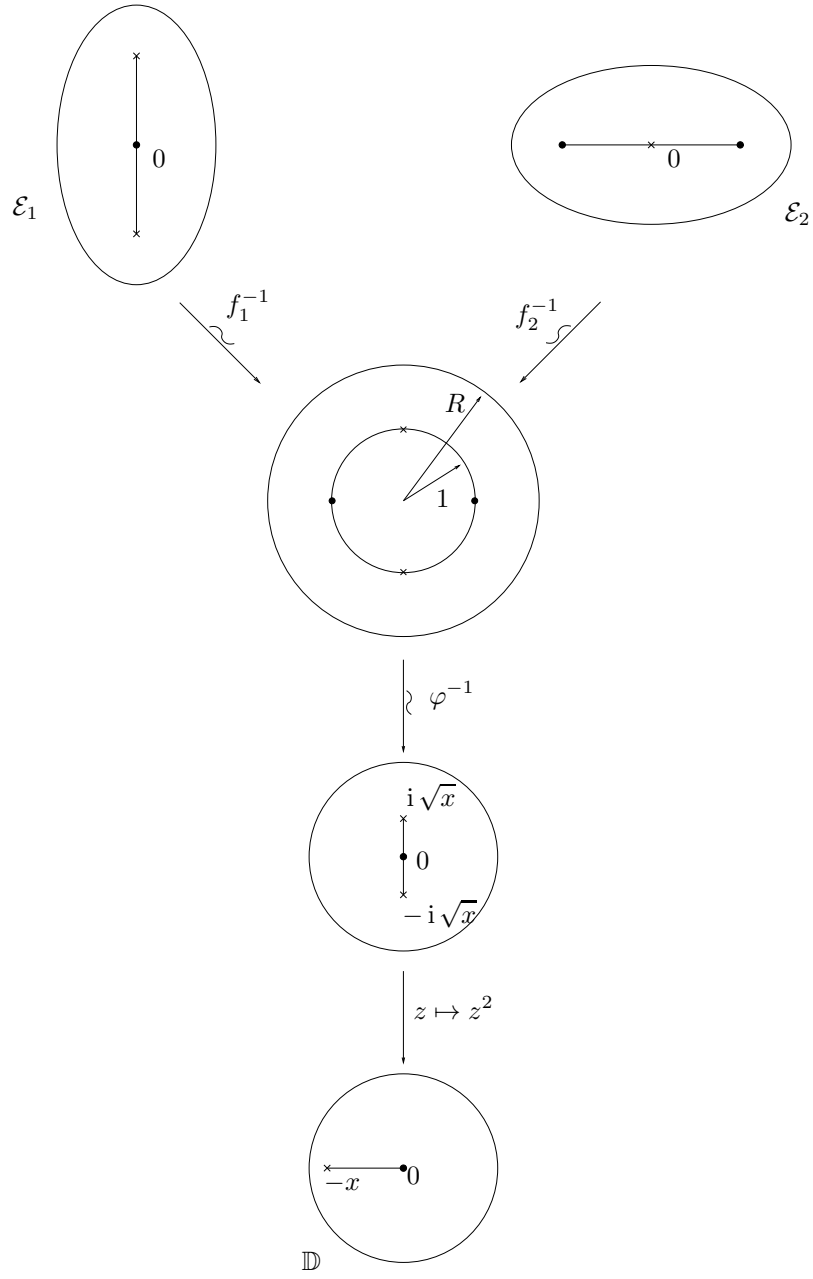


FIGURE 3. We have two different covering spaces of  $\mathbb{D} \setminus [-x, 0]$  given by  $\mathcal{E}_1$  and  $\mathcal{E}_2$ . We distinguish by crosses and points the successive inverse images of  $-x$  and  $0$ .

the boundary coincides with the restriction to the boundary of the affine map

$$\tilde{f}_0 : x + iy \mapsto \frac{R + \frac{1}{R}}{R - \frac{1}{R}}x + i \frac{R - \frac{1}{R}}{R + \frac{1}{R}}y.$$

Since this map is symmetric with respect to 0, we can conclude by construction of these two 2-sheeted ramified coverings that it descends to a q.c. mapping  $f_{(0,x)}$  with the same q.c. dilatation as  $\tilde{f}_0$ . Furthermore,  $f_{(0,x)}$  satisfies the conditions of Problem 1.1 and by using Proposition 4.2 we conclude that

$$(4.7) \quad K_f \geq \left( \frac{R^2 + 1}{R^2 - 1} \right)^2,$$

with equality if and only if  $f = f_{(0,x)}$ . We call  $f_{(0,x)}$  the *extremal map* for Problem 1.1.

It is important to note that the existence and the uniqueness of such a mapping cannot be deduced from the so-called *Teichmüller theorem*, whose the first statement can be found in [28].<sup>9</sup> Indeed, in the present case, all points (and not only a finite number) on the boundary are fixed.

However, let us observe the following interesting fact. We recall that the *Beltrami differential* associated with a q.c. mapping  $f : \mathbb{D} \rightarrow \mathbb{D}$  is an element of  $L^\infty(\mathbb{D})$  which is defined by

$$\mu_f := \frac{\partial_{\bar{z}} f}{\partial_z f}.$$

As  $f_{(0,x)} \circ p_1 = p_2 \circ \tilde{f}_0$ , we conclude by using conformality of  $p_i$  ( $i = 1, 2$ ) that

$$\mu_{f_{(0,x)}} \circ p_1 = \left( \frac{p'_1}{|p'_1|} \right)^2 \cdot k_{(0,x)};$$

where  $k_{(0,x)} = \frac{K_{f_{(0,x)}} + 1}{K_{f_{(0,x)}} - 1}$ . In other words, we have

$$(4.8) \quad \mu_{f_{(0,x)}} = k_{(0,x)} \cdot \frac{\bar{\phi}}{|\phi|},$$

where  $\phi$  is a meromorphic function on  $\mathbb{D}$  with a pole of order 1 at 0. Through Equality (4.8), the knowledgeable reader will recognize the general expression of what we call the *Teichmüller mapping*. By the way, there are works of Strebel where Equality (4.8) is a consequence of the so-called *Frame Mapping Criterion*. We refer to [26, 25]. See also [22] (p. 124). We have also to mention §159 of [29] where Teichmüller had already guessed that the extremal map satisfies Equation (4.8). In fact, Teichmüller explained that for a given homeomorphism of the disc (i.e. a condition for all boundary points) we can always extend this map to a map with the smallest q.c. dilatation and which is related to a quadratic differential by an equation of type (4.8). Note that this is at the idea of what is called the *non-reduced* Teichmüller theory and for which Problem 1.1 is an example.

Before getting further, let us note that to solve Problem 1.1, we are in a situation equivalent to the Grötzsch problem. Moreover, to solve the Grötzsch problem for two rectangles  $\mathcal{Q}_1(a_1, b_1, c_1, d_1)$  and  $\mathcal{Q}_2(a_2, b_2, c_2, d_2)$ , we only need to consider q.c. mappings which send  $a_1$  to  $a_2$ ,  $b_1$  to  $b_2$ ,  $c_1$  to  $c_2$  and  $d_1$  to  $d_2$ . With this in mind, we see that if  $f$  is a q.c. mapping from  $\mathbb{D}$  to itself sending 0 to  $-x$ , preserving the boundary and fixing 1 and  $-1$ , then its lift  $\tilde{f} : \mathcal{E}_1 \rightarrow \mathcal{E}_2$  sends the four extremal

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<sup>9</sup> Teichmüller proved in [28] the uniqueness. The existence is stated there as a “conjecture.” Teichmüller proved existence (for closed surfaces) later in [29]. We refer to the corresponding commentaries [6] and [1].

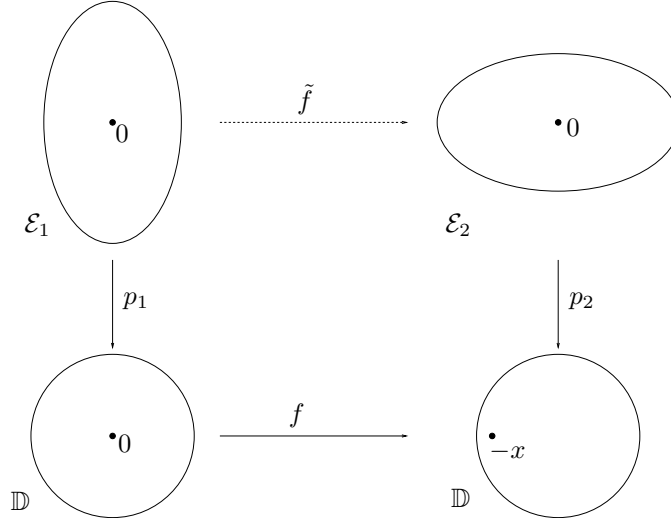


FIGURE 4. A q.c. mapping  $f$  from  $\mathbb{D}$  to  $\mathbb{D}$  such that  $f(0) = -x$  can be lifted to a q.c. mapping with the same q.c. dilatation.

points of  $\mathcal{E}_1$  to the four extremal points of  $\mathcal{E}_2$ . Thus, let us ask the following question:

**Problem 4.3.** Is it possible to find and describe the quasiconformal mapping from  $\mathbb{D}$  to itself such that

- the images of 1,  $-1$  and 0 are respectively 1,  $-1$  and  $-x$  where  $0 < x < 1$ ,
- its quasiconformal dilatation is smallest possible?

**4.3. First consequences.** We have just seen that the extremal map  $f_{(0,x)} : \mathbb{D} \rightarrow \mathbb{D}$  for Problem 1.1 has a q.c. dilatation  $K_{f_{(0,x)}}$  equal to  $\left(\frac{R^2+1}{R^2-1}\right)^2$ . Moreover, as we wrote above,  $R = \sqrt{\Phi\left(\frac{1}{x}\right)}$ , and so

$$(4.9) \quad K_{f_{(0,x)}} = \left(\frac{\Phi\left(\frac{1}{x}\right) + 1}{\Phi\left(\frac{1}{x}\right) - 1}\right)^2.$$

By using (3.2) in (4.3), we conclude that

$$(4.10) \quad K_{f_{(0,x)}} =_{0+} 1 + x + o_{0+}(x),$$

which means that

$$\frac{K_{f_{(0,x)}} - 1 - x}{x} \xrightarrow{x \rightarrow 0^+} 0.$$

On the other hand, the right hand side of (3.1) gives us

$$K_{f_{(0,x)}} > \left(1 + \frac{x}{2}\right)^2.$$

The last inequality leads us to

**Corollary 4.4.** *Let  $f : \mathbb{D} \rightarrow \mathbb{D}$  be a q.c. mapping such that  $f|_{\partial\mathbb{D}} = \text{id}_{\partial\mathbb{D}}$  and  $f(0) \in ]-1, 0]$ . Then*

$$|f(0)| \leq 2 \left( K_f^{\frac{1}{2}} - 1 \right).$$

Note that there is a small mistake in Teichmüller's paper which is considered as a "misprint" by Earle and Lakic in [7]. Indeed, Teichmüller forgot to take the power 2 in the right hand side of Relation (4.9) and so he obtained a different asymptotic behaviour in (4.10) and a different upper bound in Corollary 4.4. The same mistake appears in [9, 14].<sup>10</sup> About this mistake, we refer also to the editor footnote of [29].

We denote the hyperbolic distance<sup>11</sup> on the disc by  $d_{\mathbb{D}}(.,.)$ . According to (4.3), we can express the q.c. dilatation of  $f_{(0,x)}$  with respect to this distance by the following formula:

$$(4.11) \quad \log(K_{f_{(0,x)}}) = 2 \cdot d_{\mathbb{D}}\left(0, \frac{1}{\Phi\left(\frac{1}{x}\right)}\right).$$

## 5. SOME APPLICATIONS

In this section, we mention some applications of Teichmüller's result obtained by various authors.

**5.1. Kra's distance.** Before setting the Kra distance, we show an easy extension of Teichmüller's result. By extension, we mean to find for any distinct pair of points in  $\mathbb{D}$  the q.c. mapping from  $\mathbb{D}$  to  $\mathbb{D}$  sending one point to the other, keeping the boundary pointwise fixed and with the smallest q.c. dilatation.

Let  $z_1$  and  $z_2$  be two distinct points in  $\mathbb{D}$ . We denote by  $\varphi_{(z_1, z_2)}$  the biholomorphism of the disc which sends  $z_1$  to 0 and  $z_2$  to some point  $-x$ ,  $0 < x < 1$ . It is easy to check that the extremal map<sup>12</sup> is

$$f_{(z_1, z_2)} = \varphi_{(z_1, z_2)} \circ f_{(0, x)} \circ \varphi_{(z_1, z_2)}^{-1},$$

where  $f_{(0, x)}$  is the previous extremal map. The Beltrami differential of  $f_{(z_1, z_2)}$  is related to a meromorphic function on the disc with a pole of order 1 at  $z = z_1$  by a relation analogous to (4.8). Strebel calls in [26] such a mapping the *Teichmüller shift*.

Furthermore,

$$(5.12) \quad d : (z_1, z_2) \in \mathbb{D} \times \mathbb{D} \mapsto \frac{1}{2} \log \left( K_{f_{(z_1, z_2)}} \right)$$

defines a new distance on the disc. Moreover, as Kra observed in [14],  $d$  is a complete metric.

We have all the ingredients to define the Kra distance. Let  $S$  be a hyperbolic Riemann surface of finite type  $(g, n)$ , where  $g$  is the genus and  $n$  the number of punctures. We recall that a hyperbolic Riemann surface is a Riemann surface whose

<sup>10</sup> I. Kra informed the author of the way he discovered this mistake. Kra needed some consequences of Teichmüller results to write [14] and he used for this the Gehring paper [9]. Gehring found this error only after publishing [9] and when he knew that Kra used his paper, he informed him.

<sup>11</sup>We use the metric with constant curvature  $-1$ .

<sup>12</sup>We mean here the map with the smallest q.c. dilatation, sending  $z_1$  to  $z_2$  and keeping the boundary pointwise fixed.

universal cover is the unit disc. This implies in particular that  $S$  carries a hyperbolic metric. Kra defined in [14] a new distance on  $S$  as follows. For any two points  $x$  and  $y$  in  $S$ , he sets

$$(5.13) \quad d_{\text{Kr}}(x, y) := \frac{1}{2} \log \inf_f K_f,$$

where the infimum is taken over all q.c. mappings<sup>13</sup>  $f$  isotopic to the identity mapping and sending  $x$  to  $y$ . This distance is now called the *Kra distance*.<sup>14</sup> From a compactness property of q.c. mappings we know that there always exists a q.c. mapping which attains the infimum in (5.13). Kra obtained the uniqueness of such a mapping if  $x$  and  $y$  are close enough for the hyperbolic metric on  $S$  (see Proposition 6. in [14]). Furthermore, he showed that  $d_{\text{Kr}}$  is equivalent to the hyperbolic metric but not proportional to it unless  $S$  is the thrice-punctured sphere. In this exceptional case the two metrics coincide. It is of interest to note that the idea of Kra's distance already exists in [28]. Indeed, Teichmüller introduced such a distance and he showed in §27 that it coincides with the hyperbolic distance in the case of the thrice-punctured sphere. In the same paper, Teichmüller explained in §160 that up to a condition,  $S$  equipped with  $d_{\text{Kr}}$  is a *Finsler manifold*.

**5.2. About a problem of Gehring and a little bit more.** The *Gehring problem*, which could be seen as a dual of Problem 1.1, is the following. Given  $K > 1$ , we want to describe the value

$$(5.14) \quad h_{\mathbb{D}}(K) := \sup \{d_{\mathbb{D}}(z, f(z)) \mid z \in \mathbb{D} \text{ and } f \in \mathcal{Q}_{\mathbb{D}}(K)\};$$

where  $\mathcal{Q}_{\mathbb{D}}(K)$  denotes the set of all  $K$ -q.c. mappings from  $\mathbb{D}$  to  $\mathbb{D}$  which hold the boundary pointwise fixed.

This problem can be addressed for any planar domain  $\Omega$  with at least 3 boundary points. Indeed, for such a domain, we know that the universal cover is the unit disc, so by pushing forward the hyperbolic metric on the disc, we can define a hyperbolic metric of constant curvature  $-1$  on  $\Omega$ . We denote it by  $d_{\Omega}(\cdot, \cdot)$  and we may want to determine the value of (5.14) by considering  $d_{\Omega}$  instead of  $d_{\mathbb{D}}$ .

Krzyż gave a value for (5.14) in [15]. He proved that there exists  $z_0 \in \mathbb{D}$  and  $f_K \in \mathcal{Q}_{\mathbb{D}}(K)$  such that

$$h_{\mathbb{D}}(K) = d_{\mathbb{D}}(z_0, f_K(z_0)).$$

He gave a precise value of  $h_{\mathbb{D}}(K)$  and he showed, by using an analogue of Corollary 4.4, that  $f_K$  is the extremal map with respect to the Teichmüller problem (i.e. the extension of Problem 1.1 where the pair of points is  $(z_0, f_K(z_0))$ ).

Later, Solynin and Vuorinen showed in [24] that the supremum of (5.14) is attained for a unique map, the map described above.

The Gehring problem can also be addressed for domains in  $\mathbb{R}^n$ , where  $n > 2$ . See for example [31] and [18]. These two papers are related to a paper of Martin [19]. Furthermore, Martin worked in [20] on an extremal problem close to Teichmüller's one. To be more precise, he considered for  $0 \leq x < 1$ , the value

<sup>13</sup>A q.c. mapping on a Riemann surface is a mapping whose a lift to the universal cover is a q.c. mapping and the q.c. dilatation is the q.c. dilatation of this lift.

<sup>14</sup>It seems that this name appears for the first time in [23].

$$(5.15) \quad \inf \left\{ \frac{1}{\pi} \iint_{\mathbb{D}} K_f(z) \frac{1}{2} dz \wedge d\bar{z} \mid f \text{ is q.c.}, f(0) = -x \text{ and } f|_{\partial\mathbb{D}} = \text{id}_{\partial\mathbb{D}} \right\}.$$

He showed that if  $x > 0$ , the infimum in (5.15) cannot be attained by a q.c. mapping.

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